

A DC programming approach for the constrained two-dimensional non-guillotine cutting problem^{*}

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Abstract

We investigate a new application of DC (Difference of Convex functions) programming and DCA (DC Algorithm) in solving the constrained two-dimensional non-guillotine cutting problem. This problem consists of cutting a number of rectangular pieces from a large rectangular object. The cuts are done under some constraints and the objective is to maximize the total value of the pieces cut. We reformulate this problem as a DC program and solve it by DCA. The performance of the approach is compared with the standard solver CPLEX.

Key words: DC Programming, DCA, Constrained two-dimensional non-guillotine cutting.

1 Introduction

The field of combinatorial optimization involves many challenging problems. Different practical applications of these problems, motivates the researchers to develop new methods in order to solve them as efficiently as possible. One of the important classes of combinatorial optimization problems is the class of the cutting and packing problems. The constrained two-dimensional non-guillotine cutting problem (NGC) is one of the cutting and packing problems that have been studied by several researchers ([1,2,12,18]). The constrained two-dimensional non-guillotine cutting problem consists of cutting a number of rectangular pieces from a large rectangular object. The cuts are done under some constraints and the objective is to maximize the total value of the pieces cut.

This problem arises in several practical applications, such as cutting the steel or glass plates into required sizes, cutting the wood sheets to make furniture etc. [1]. The optimal solution of this problem minimizes the amount of wastes produced (see Fig. 1).

In this study, we consider the constrained two-dimensional non-guillotine cutting problem and we investigate a deterministic approach based on DC programming techniques to solve it. The particular interest of the work is in the design of the algorithms called DCA (DC Algorithm). This approach is a local deterministic method based on DC (Difference of Convex functions) programming. The DC Algorithm

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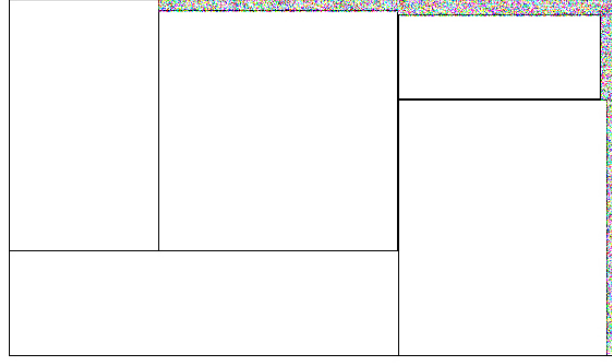


Fig. 1. A two-dimensional non-guillotine cutting pattern in which the produced wastes are minimized

(DCA) was first introduced, in its preliminary form, by Pham Dinh Tao in 1985 and has been extensively developed since 1994 by Le Thi Hoai An and Pham Dinh Tao. It becomes now classic and popular. DCA has been successfully applied to many large-scale (smooth or nonsmooth) non-convex programs in various domains of applied sciences, for example *tomography* [19], *finance* [8,9,11], and *machine learning* [10,13,17]. Numerical experiments show that DCA is in many cases more robust and efficient than standard methods (see e.g. [3,4,6,8,9,10,11,13,14,15,19] and reference therein).

In this work, we first formulate the underlying optimization models in the form of a DC programs in which a DC function is minimized over a closed convex set. Then, DCA is used to solve it. Computational experiences, performed over standard benchmark problems, show that DCA is quite efficient for solving the constrained two-dimensional non-guillotine cutting problem and compares favorably with the standard solver ILOG CPLEX.

The structure of the paper is as follows. The constrained two-dimensional non-guillotine cutting problem is reviewed in the section 2. In section 3 we give an outline of general DC programs and DCA. The DC program for the constrained two-dimensional non-guillotine cutting problem is discussed in section 4. The computational experiences are reported in section 5 and the last section includes some conclusions.

2 The constrained two-dimensional non-guillotine cutting problem

Consider a large rectangular object. In the constrained two-dimensional non-guillotine cutting problem, we are interested in cutting some smaller rectangular pieces from this large rectangular object. We suppose that the cuts are done in a way that the edges of the smaller pieces are parallel to the edges of the large rectangular object and there must be no overlapping between the pieces cut. Furthermore, we suppose that the orientation of each of the smaller objects is known in advance and a limited number of the small pieces is available. There is a positive integer number associated to each rectangle that indicates its value. The objective of the constrained two-dimensional non-guillotine cutting problem is to cut the smaller pieces from the large rectangle in a way that the total value of the pieces cut be maximized.

The constrained two-dimensional non-guillotine cutting problem is known to be an NP-Complete problem, so it is very difficult to solve the problem efficiently. The problem has been the object of several articles. The articles [1], [2], and [12] summarize some of the works done on this problem.

2.1 Mathematical Formulation

In this section we present the mathematical formulation of the constrained two-dimensional non-guillotine cutting problem. We use the notations used by Napoleao et al. [12]. This formulation concerns a 0 – 1 linear programming model that has been already presented in [1]. There is also a nonlinear programming formulation that has been introduced in [2].

Suppose that m types of pieces are available. For each type i of the pieces, we know the characteristics of the object, such as its length and width (l_i, w_i) , its value v_i and we know that only a limited number of the piece i is available, which is noted by b_i . The pieces must be cut from a large rectangular object

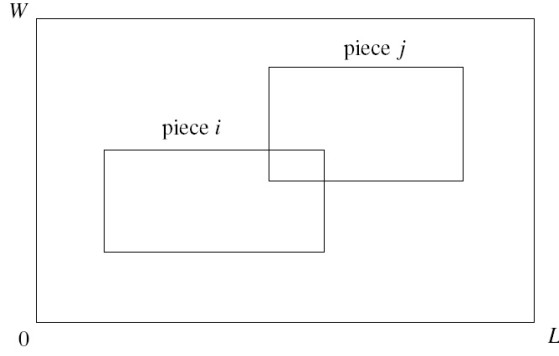


Fig. 2. Overlapping of the pieces i and j .

with the length L and the width W . We use a binary variable x_{ipq} to say whether or not the piece i can be cut orthogonally from the large object at the position (p, q) :

$$x_{ipq} = \begin{cases} 1, & \text{if a piece of type } i \text{ is allocated at position } (p, q), \\ 0, & \text{otherwise.} \end{cases}$$

One can assume that p and q belong, respectively, to the following sets $([1, 12])$:

$$P := \left\{ p : p = \sum_{i=1}^m \alpha_i l_i, p \leq L - \min\{l_i, i = 1, \dots, m\}, \alpha_i \geq 0, \alpha_i \in \mathbf{Z} \right\}, \quad (1)$$

$$Q := \left\{ q : q = \sum_{i=1}^m \beta_i w_i, q \leq W - \min\{w_i, i = 1, \dots, m\}, \beta_i \geq 0, \beta_i \in \mathbf{Z} \right\}. \quad (2)$$

The cuts cannot pass over the edges of the large rectangle, so we need to take into account the following sets:

$$P_i := \{p : p \in P, p \leq L - l_i\}, \quad (3)$$

$$Q_i := \{q : q \in Q, q \leq W - w_i\}. \quad (4)$$

The coefficients a_{ipqrs} are defined as follows in order to prevent the interposition of the pieces cut (see Fig. 2):

$$a_{ipqrs} = \begin{cases} 1, & \text{if } p \leq r \leq p + l_i - 1 \text{ and } q \leq s \leq q + w_i - 1, \\ 0, & \text{otherwise.} \end{cases}$$

The a_{ipqrs} are defined for each piece $i = 1, \dots, m$, and for each coordinates (p, q) as well as (r, s) .

Using these notations, we can now present the mathematical formulation of the constrained two-dimensional non-guillotine cutting problem:

(NGC) :

$$\begin{aligned} & \max \sum_{i=1}^m \sum_{p \in P_i} \sum_{q \in Q_i} v_i x_{ipq} \\ & \text{s.t.} \\ & \sum_{i=1}^m \sum_{p \in P_i} \sum_{q \in Q_i} a_{ipqrs} x_{ipq} \leq 1 : \forall r \in P, s \in Q, \\ & \sum_{p \in P_i} \sum_{q \in Q_i} x_{ipq} \leq b_i : i = 1, \dots, m, \\ & x_{ipq} \in \{0, 1\} : \forall i, p, q. \end{aligned}$$

By solving this problem, one maximizes the total value associated to the all pieces cut from the large rectangular object.

The model includes $(|P||Q| + m)$ constraints and $n := (\sum_{i=1}^m \sum_{p \in P_i} \sum_{q \in Q_i} 1)$ variables. It means that the problem takes a huge dimension even by moderate values of $|P_i|$ or $|Q_i|$ ($i = 1, \dots, m$).

3 General DC programs and DCA

Let $\Gamma_0(\mathbb{R}^n)$ denotes the convex cone of all lower semi-continuous proper convex functions on \mathbb{R}^n . Consider the following primal DC program

$$(P_{dc}) \quad \beta_p = \inf\{F(x) := g(x) - h(x) : x \in \mathbb{R}^n\},$$

where $g, h \in \Gamma_0(\mathbb{R}^n)$.

A DC program (P_{dc}) is called a DC polyhedral program when either g or h is a polyhedral convex function (i.e., the pointwise supremum of a finite collection of affine functions). Note that a polyhedral convex function is almost always differentiable, say, it is differentiable everywhere except on a set of measure zero.

Let C be a nonempty closed convex set. Then, the problem

$$\inf\{f(x) := g(x) - h(x) : x \in C\}, \tag{5}$$

can be transformed into an unconstrained DC program by using the indicator function of C , i.e.,

$$\inf\{f(x) := \phi(x) - h(x) : x \in \mathbb{R}^n\}, \tag{6}$$

where $\phi := g + \chi_C$ is in $\Gamma_0(\mathbb{R}^n)$.

Let $g^*(y) := \sup\{\langle x, y \rangle - g(x) : x \in \mathbb{R}^n\}$ be the conjugate function of g . Then, the following program is called the dual program of (P_{dc}) :

$$(D_{dc}) \quad \beta_d = \inf\{h^*(y) - g^*(y) : y \in \mathbb{R}^n\}. \tag{7}$$

Under the natural convention in DC programming that is $+\infty - (+\infty) = +\infty$, and by using the fact that every function $h \in \Gamma_0(\mathbb{R}^n)$ is characterized as a pointwise supremum of a collection of affine functions, say

$$h(x) := \sup\{\langle x, y \rangle - h^*(y) : y \in \mathbb{R}^n\},$$

it can be proved that $\beta_p = \beta_d$ [15]. There is a perfect symmetry between primal and dual DC programs, that is the dual of (D_{dc}) is (P_{dc}) .

Recall that, for $\theta \in \Gamma_0(\mathbb{R}^n)$ and $x_0 \in \text{dom } \theta := \{x \in \mathbb{R}^n | \theta(x_0) < +\infty\}$, the subdifferential of θ at x_0 , denoted $\partial\theta(x_0)$, is defined as

$$\partial\theta(x_0) := \{y \in \mathbb{R}^n : \theta(x) \geq \theta(x_0) + \langle x - x_0, y \rangle, \forall x \in \mathbb{R}^n\} \tag{8}$$

which is a closed convex set in \mathbb{R}^n . It generalizes the derivative in the sense that θ is differentiable at x_0 if and only if $\partial\theta(x_0)$ is reduced to a singleton which is exactly $\{\nabla\theta(x_0)\}$.

The necessary local optimality condition for the primal DC program, (P_{dc}) , is

$$\partial h(x^*) \subset \partial g(x^*). \tag{9}$$

The condition (9) is also sufficient for many important classes of DC programs, for example, for DC polyhedral programs, or when function f is locally convex at x^* [6,14].

A point x^* satisfies the generalized Kuhn-Tucker condition

$$\partial h(x^*) \cap \partial g(x^*) \neq \emptyset \quad (10)$$

is called a critical point of $g - h$. It follows that if h is a polyhedral convex function, then a critical point of $g - h$ is almost always a local solution to (P_{dc}) .

The transportation of global solutions between (P_{dc}) and (D_{dc}) is expressed by [4,6,14,15]:

Property 1:

$$[\cup_{y^* \in \mathcal{D}} \partial g^*(y^*)] \subset \mathcal{P}, \quad [\cup_{x^* \in \mathcal{P}} \partial h(x^*)] \subset \mathcal{D}, \quad (11)$$

where \mathcal{P} and \mathcal{D} denote the solution sets of (P_{dc}) and (D_{dc}) respectively.

Under certain technical conditions, this property also holds for the local solutions of (P_{dc}) and (D_{dc}) [4,6,14,15]. For example the following result holds:

Property 2: Let x^* be a local solution to (P_{dc}) and let $y^* \in \partial h(x^*)$. If g^* is differentiable at y^* then y^* is a local solution to (D_{dc}) . Similarly, let y^* be a local solution to (D_{dc}) and let $x^* \in \partial g^*(y^*)$. If h is differentiable at x^* then x^* is a local solution to (P_{dc}) .

Based on local optimality conditions and duality in DC programming, the DC Algorithm (DCA) consists in constructing two sequences $\{x^l\}$ and $\{y^l\}$ of trial solutions for the primal and dual programs, respectively, such that the sequences $\{g(x^l) - h(x^l)\}$ and $\{h^*(y^l) - g^*(y^l)\}$ are decreasing, and $\{x^l\}$ (resp. $\{y^l\}$) converges to a primal feasible solutions \tilde{x} (resp. a dual feasible solution \tilde{y}) satisfying the local optimality condition and

$$\tilde{x} \in \partial g^*(\tilde{y}), \quad \tilde{y} \in \partial h(\tilde{x}). \quad (12)$$

DCA then yields the next simple scheme:

$$y^l \in \partial h(x^l); \quad x^{l+1} \in \partial g^*(y^l). \quad (13)$$

In other words, these two sequences $\{x^l\}$ and $\{y^l\}$ are determined in the way that x^{l+1} and y^{l+1} are solutions of the convex primal program (P_l) and dual program (D_{l+1}) , respectively. These are defined as

$$(P_l) \quad \inf \{g(x) - h(x^l) - \langle x - x^l, y^l \rangle : x \in \mathbb{R}^n\}, \quad (14)$$

$$(D_{l+1}) \quad \inf \{h^*(y) - g^*(y^l) - \langle y - y^l, x^{l+1} \rangle : y \in \mathbb{R}^n\}. \quad (15)$$

At each iteration, the DCA performs a double linearization with the use of the subgradients of h and g^* . In fact, in each iteration, one replaces in the primal DC program, (P_{dc}) , the second component h by its affine minorization $h_l(x) := h(x^l) + \langle x - x^l, y^l \rangle$ to construct the convex program (P_l) whose solution set is nothing but $\partial g^*(y^l)$. Likewise, the second DC component g^* of the dual DC program, (D_{dc}) , is replaced by its affine minorization $g_l^*(y) := g^*(y^l) + \langle y - y^l, x^{l+1} \rangle$ to obtain the convex program (D_{l+1}) whose $\partial h(x^{l+1})$ is the solution set. Hence DCA works with the convex DC components g and h but not with the DC function f itself. Moreover, a DC function f has infinitely many DC decompositions which have crucial impacts on the performance of the DCA in terms of speed of convergence, robustness, efficiency, and globality of computed solutions. Convergence properties of the DCA and its theoretical basis are described in [4,6,14,15]. However, it is worthwhile to summarize the following properties for the sake of completeness :

- DCA is a descent method (*without line search*). The sequences $\{g(x^l) - h(x^l)\}$ and $\{h^*(y^l) - g^*(y^l)\}$ are decreasing such that

$$g(x^{l+1}) - h(x^{l+1}) \leq h^*(y^l) - g^*(y^l) \leq g(x^l) - h(x^l).$$

- If $g(x^{l+1}) - h(x^{l+1}) = g(x^l) - h(x^l)$, then x^l is a critical point of $g - h$ and y^l is a critical point of $h^* - g^*$. In this case, DCA terminates at l^{th} iteration.

- If the optimal value β_p of problem (P_{dc}) is finite and the infinite sequences $\{x^l\}$ and $\{y^l\}$ are bounded, then every limit point \tilde{x} (resp. \tilde{y}) of the sequence $\{x^l\}$ (resp. $\{y^l\}$) is a critical point of $g - h$ (resp. $h^* - g^*$).
- DCA has linear convergence for general DC programs. For polyhedral DC programs the sequences $\{x^l\}$ and $\{y^l\}$ contain finitely many elements and the algorithm converges to a solution in a finite number of iterations.

4 DC program for the constrained two-dimensional non-guillotine cutting problem

We consider a new approach based on DC programming and DCA to solve the constrained two-dimensional non-guillotine cutting problem. The DCA requires a reformulation of the problem so that the objective function be represented by the difference of two convex functions. Then the original problem becomes a DC program in which the DC function is minimized over a convex set. In this section, we introduce the corresponding DC reformulations of the NGC problem and then present a DCA to solve the corresponding DC program.

Using the exact penalty result presented in [7], we will formulate (NGC) in the form of a DC minimization problem with linear constraints which is consequently a DC program. Let

$$A := \{\mathbf{x} \in [0, 1]^n : \sum_{i=1}^m \sum_{p \in P_i} \sum_{q \in Q_i} a_{ipqrs} x_{ipq} \leq 1 : \forall r \in P, s \in Q, \sum_{p \in P_i} \sum_{q \in Q_i} x_{ipq} \leq b_i : i = 1, \dots, m\}.$$

Let $\alpha(\mathbf{x})$ be the concave function defined as follows

$$\alpha(\mathbf{x}) := \sum_{i=1}^m \sum_{p \in P_i} \sum_{q \in Q_i} x_{ipq} (1 - x_{ipq}),$$

The concave function $\alpha(x)$ is non-negative on A hence (NGC) can be re-written as follows

$$(\text{NGC} - 2) : \quad \min \left\{ - \sum_{i=1}^m \sum_{p \in P_i} \sum_{q \in Q_i} v_i x_{ipq} : \alpha(\mathbf{x}) \leq 0, \mathbf{x} \in A \right\}.$$

Since the objective function is linear, A is a bounded polyhedral convex set, and the concave function $\alpha(x)$ is non-negative on A ; according to [7], there is $t_0 \geq 0$ such that for any $t > t_0$, the program (NGC - 2) is equivalent to

$$(\text{NGC} - 3) : \quad \min \left\{ F(\mathbf{x}) := - \sum_{i=1}^m \sum_{p \in P_i} \sum_{q \in Q_i} v_i x_{ipq} + t\alpha(\mathbf{x}) : \mathbf{x} \in A \right\}.$$

The function F is concave in variables \mathbf{x} ; consequently it is a DC function. A natural DC formulation of the problem (NGC - 3) is

$$(\text{NGC-DC}) : \quad \min \{ F(\mathbf{x}) := g(\mathbf{x}) - h(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^n \},$$

where

$$g(\mathbf{x}) = - \sum_{i=1}^m \sum_{p \in P_i} \sum_{q \in Q_i} v_i x_{ipq} + \chi_A(\mathbf{x})$$

and

$$h(\mathbf{x}) = t \sum_{i=1}^m \sum_{p \in P_i} \sum_{q \in Q_i} x_{ipq} (x_{ipq} - 1).$$

Here χ_A is the indicator function on A , i.e. $\chi_A(\mathbf{x}) = 0$ if $(\mathbf{x}) \in A$ and $+\infty$ otherwise.

4.1 DCA for solving (NGC-DC)

According to the general framework of DCA, we first need computing a sub-gradient of the function $h(\mathbf{x})$ defined by $h(\mathbf{x}) = t \sum_{i=1}^m \sum_{p \in P_i} \sum_{q \in Q_i} x_{ipq}(x_{ipq} - 1)$. From the definition of $h(\mathbf{x})$ we have

$$\mathbf{y}^k \in \partial h(\mathbf{x}^k) \Leftrightarrow y_{ipq}^k := t(2x_{ipq}^k - 1), \quad (16)$$

for $i = 1, \dots, m$, $p \in P_i$, and $q \in Q_i$.

Secondly, we need to compute an optimal solution of the following linear program

$$\min \left\{ - \sum_{i=1}^m \sum_{p \in P_i} \sum_{q \in Q_i} v_i x_{ipq} - \langle \mathbf{x}, \mathbf{y}^k \rangle : \mathbf{x} \in A \right\} \quad (17)$$

that will be \mathbf{x}^{k+1} . To sum up, the DCA applied to (NGC-DC) can be described as follows.

Algorithm DCA

(1) **Initialization:** Choose $\mathbf{x}^0 \in \mathbb{R}^n$, $\epsilon > 0$, $t > 0$, and set $k = 0$.

(2) **Iteration:**

Set $y_{ipq}^k := t(2x_{ipq}^k - 1)$ for $i = 1, \dots, m$, $p \in P_i$, and $q \in Q_i$.

Solve the following linear program

$$\min \left\{ - \sum_{i=1}^m \sum_{p \in P_i} \sum_{q \in Q_i} v_i x_{ipq} - \langle \mathbf{x}, \mathbf{y}^k \rangle : \mathbf{x} \in A \right\}$$

to obtain \mathbf{x}^{k+1} .

(3) If $\|\mathbf{x}^{k+1} - \mathbf{x}^k\| \leq \epsilon$ then **STOP** and take \mathbf{x}^{k+1} as an optimal solution, otherwise set $k = k + 1$ and go to step 2.

Finding a good initial point for DCA

In fact, one of the key questions in DCA is how to find a good initial solution for it. In this work, we solved a relaxed form of the (NGC-DC) program to find a good initial solution. To this aim, we relax the following constraints from the (NGC-DC) program:

$$\sum_{i=1}^m \sum_{p \in P_i} \sum_{q \in Q_i} a_{ipqrs} x_{ipq} \leq 1 \quad : \quad \forall r \in P, s \in Q, \quad (18)$$

and we add them to the objective function of the (NGC-DC) program. In order to penalize any violation of the relaxed constraints, a sufficiently large positive real number u is used as the penalty parameter.

Let $\beta(\mathbf{x})$ be the function defined by

$$\beta(\mathbf{x}) := \max_{r \in P, s \in Q} \left(0, \left(\sum_{i=1}^m \sum_{p \in P_i} \sum_{q \in Q_i} a_{ipqrs} x_{ipq} \right) - 1 \right),$$

and define the convex set B as follows

$$B = \left\{ \mathbf{x} \in \mathbb{R}^n : \sum_{p \in P_i} \sum_{q \in Q_i} x_{ipq} \leq b_i, 0 \leq x_{ipq} \leq 1, i = 1, \dots, m \right\}.$$

Using the function $\beta(\mathbf{x})$ and the parameter u for relaxing the constraints (18) we obtain the following program

$$\min \left\{ G(\mathbf{x}) := - \sum_{i=1}^m \sum_{p \in P_i} \sum_{q \in Q_i} v_i x_{ipq} + t\alpha(\mathbf{x}) + u\beta(\mathbf{x}) : \mathbf{x} \in B \right\}. \quad (19)$$

$G(\mathbf{x})$ is a DC function and a DC formulation of (19) can be

$$\min \{ G(\mathbf{x}) := \varphi(\mathbf{x}) - \phi(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^n \}, \quad (20)$$

where

$$\varphi(\mathbf{x}) = - \sum_{i=1}^m \sum_{p \in P_i} \sum_{q \in Q_i} v_i x_{ipq} + \chi_B(\mathbf{x})$$

and

$$\phi(\mathbf{x}) = t \sum_{i=1}^m \sum_{p \in P_i} \sum_{q \in Q_i} x_{ipq} (x_{ipq} - 1) - u \cdot \max_{r,s} \left(0, \left(\sum_{i=1}^m \sum_{p \in P_i} \sum_{q \in Q_i} a_{ipqr} x_{ipq} \right) - 1 \right).$$

Here χ_B is the indicator function on B .

The solution of (20) is used as the initial point for **Algorithm DCA**. The solution of (20) may not be feasible to (NGC-DC), but we need just one iteration of DCA to obtain a feasible solution of (NGC-DC) and all the other iterations of DCA will improve the solution.

In fact, we have tested DCA from different initial points, some of them are:

- The point obtained by the above procedure;
- $\mathbf{x} = (0, \dots, 0)$;
- $\mathbf{x} = (1, \dots, 1)$;
- The optimal solution of the relaxed (NGC) problem obtained by replacing the binary constraints $x_{ipq} \in \{0, 1\}$ by $0 \leq x_{ipq} \leq 1$ for all i, p, q .

According to our experiments the initial point provided by the first procedure is the best.

Since (20) is a DC program, we use again DCA for solving it. The DCA applied on this problem is described as follows:

Algorithm Initial-DCA

(1) **Initialization:** Choose $\mathbf{x}^0 \in \mathbb{R}^n$, $\epsilon > 0$, $t > 0$, $u > 0$, and set $k = 0$;

(2) **Iteration:**

Set

$$y_{ipq}^k := \begin{cases} t(2x_{ipq}^k - 1), & \text{if all } (r, s) \text{ constraints are satisfied at the point } \mathbf{x}^k, \\ t(2x_{ipq}^k - 1) - u(a_{ipqr}), & \text{otherwise: for some } (r, s), \end{cases}$$

for $i = 1, \dots, m$, $p \in P_i$, and $q \in Q_i$.

Solve the following linear program to obtain \mathbf{x}^{k+1} :

$$\min \left\{ - \sum_{i=1}^m \sum_{p \in P_i} \sum_{q \in Q_i} v_i x_{ipq} - \langle \mathbf{x}, \mathbf{y}^k \rangle : \mathbf{x} \in B \right\}$$

(3) If $\|\mathbf{x}^{k+1} - \mathbf{x}^k\| \leq \epsilon$ then **STOP** and take \mathbf{x}^{k+1} as an optimal solution to (20), otherwise set $k = k + 1$ and go to step 2.

Clearly we need an initial point to start **Algorithm Initial-DCA**. This time we use the solution of the linear program obtained by relaxation of the binary constraints in (NGC) (the binary constraints $x_{ipq} \in \{0, 1\}$ are replaced by $0 \leq x_{ipq} \leq 1$ for all i, p, q).

5 Computational experiences

Some experiments have been carried out to evaluate the quality of the solutions provided by the proposed algorithm. The solutions have been compared with the results given by the standard solver CPLEX version 11.2. This solver have been used to solve the binary programming (NGC) problem.

The experiments have been performed over 12 benchmark test problems. The test problems, taken from literature [1,2,12,18], are also available through:

<http://people.brunel.ac.uk/~mastjjb/jeb/orlib/ngcutinfo.html>.

The algorithms are coded in C++ and run on a Pentium IV 3.0 *GHz* with 1*GB* *RAM*. The precision ε has been set equal to 10^{-6} for DCA and a CPU time limit of 60 seconds has been considered for the solver CPLEX. We used the solver CPLEX version 11.2 in order to solve the linear sub-problems generated at each iteration of DCA.

Table 1

Results of the experiments carried out over 12 benchmark test problems.

	(L;W)	$m- P - Q $	N. Var.	N. Con.	CPLEX		DCA				
					CPU	Opt. val.	CPU	Opt. val.	iter.	t	u
1	(10;10)	5 - 8 - 6	60	51	0.140	164	0.062	146	4	30	10
2	(10;10)	7 - 10 - 10	250	107	0.891	230	0.156	212	5	30	30
3	(10;10)	10 - 10 - 10	411	110	0.281	247	0.156	242	4	20	5
4	(15;10)	5 - 3 - 10	68	35	0.031	268	0.063	268	4	25	25
5	(15;10)	7 - 10 - 10	154	107	0.047	358	0.329	358	23	20	80
6	(15;10)	10 - 13 - 10	552	140	12.094	289	1.019	283	26	25	200
7	(20;20)	5 - 20 - 20	763	405	0.110	430	0.328	404	4	50	100
8	(20;20)	7 - 6 - 20	343	127	9.719	834	0.234	828	6	50	100
9	(20;20)	10 - 20 - 18	1413	370	3.094	924	2.531	924	8	5	5
10	(30;30)	5 - 30 - 26	363	785	4.860	1452	0.562	1452	4	100	100
11	(30;30)	7 - 21 - 27	1120	574	60.266	1688	1.094	1688	4	100	100
12	(70;40)	20 - 45 - 18	3657	830	60.312	2726	7.204	2568	4	180	10

The results are presented in Table 1. The table contains some information about the test problems: the length (L) and the width (W) of the large rectangular object, the number of the types of the pieces to be cut and the number of the elements in the sets P and Q . The table contains also some information about the (NGC) problems corresponding to each test problem: number of the variables (N. Var.) and number of the constraints (N. Con.). In this table, the CPU time (CPU) in second, the best optimal value (Opt. val.) of the solver CPLEX and also DCA, the number of DCA iterations (iter.), and the values of the penalty parameters (t and u) are presented.

The proposed approach has presented very satisfactory results in comparison to the solver CPLEX. The solver CPLEX solves efficiently some first test problems that are small size problems, but CPLEX needs more time to solve the others. Specially, we consider the test problems number 6 and 8 – 12. For these problems, DCA has a better performance. The results are particularly interesting for the test problems number 10 and 11, for which the proposed DCA method gives the same solutions as CPLEX, but in a significantly shorter CPU time.

6 Conclusion

In this paper, we present a new approach based on DC programming and DCA to solve the constrained two-dimensional non-guillotine cutting problem. We saw that DCA outperforms in some cases the commercial solver CPLEX.

The computational results suggest to us extending the numerical experiments in higher dimensions, and combining DCA and Branch-and-Bound algorithms for globally solving the constrained two-dimensional non-guillotine cutting problem. Work in these directions is currently in progress.

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